Modern approaches to quantum gravity

Homework 6 Fall 2025

1. A stack of strings

(a) State the Polyakov and Nambu-Goto actions for a string coupled to a D-dimensional background metric $G_{\mu\nu}$, $S_{\text{Poly}}[G]$ and $S_{NG}[G]$. Assume that no background $B-$field or dilaton Φ are present.

(b) The string frame metric produced by N infinite static strings, i.e. strings, lying in the $(X^0, X^1) \equiv (t, x)$ plane is

$$ds^2 = f(r)^{-1}(-dt^2 + dx^2) + d\vec{X} \cdot d\vec{X} \tag{1}$$

where $\vec{X} = (X^2, ..., X^{D-1})$ labels the directions transverse to the strings and

$$f(r) = 1 + g_s^2 N \left(\frac{l_s}{r}\right)^{D-4}, \qquad r^2 \equiv \vec{X} \cdot \vec{X}. \tag{2}$$

Consider one further infinite probe string in this background, lying parallel to the others, separated by a distance r.
Write down the Nambu-Goto action describing the motion of this string. Show that in static gauge $t = \tau$ and $x = \sigma$, the low-energy excitations of the string are governed by the effective action

$$L \approx \frac{1}{2\pi\alpha'} \int dt dx \Big[-f(r)^{-1} + \frac{1}{2}\Big(\frac{d\vec{X}}{dt} \cdot \frac{d\vec{X}}{dt} - \frac{d\vec{X}}{dx} \cdot \frac{d\vec{X}}{dx}\Big) + ...\Big] \tag{3}$$

Find the equations of motion of the string and interpret the result.

If the string is also coupled to an external $B-$field, the worldsheet action is modified by adding

$$S = S[G] + \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g}\; B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta} \tag{4}$$

where we introduced the antisymmetric tensor density $\epsilon^{\alpha\beta}$, normalised so that $\sqrt{g}\epsilon^{01} = 1$.

(c) In the setup from the previous point, the N strings produce a background $B-$field with a single independent component

$$B_{01} = f(r)^{-1} - 1 \tag{5}$$

Show that the probe string feels no static force.

2. The different regimes of the Dp-brane gravity description

Before tackling this exercise, let us give a reminder on the different parameters of string theory. The Regge slope $\alpha' = 1/M_s^2 = \ell_s^2$ sets the scale at which string states become important. The string coupling is $e^{\phi} = g_s$, and it relates to the Planck mass as $g_s \sim M_s^4/M_p^4$, thus g_s controls the size of perturbative quantum gravity corrections.
Let us consider the 10-dimensional supergravity solution describing N coincident extremal Dp-branes in the string frame

$$ds^2 = f_p^{-1/2}(-dt^2 + dx_1^2 + ...dx_p^2) + f_p^{1/2}(dx_{p+1}^2 + ... + dx_9^2) \tag{6}$$

$$e^{-2(\phi-\phi_\infty)} = f_p^{(p-3)/2} \tag{7}$$

where

$$\alpha'^2 f_p = \alpha'^2 + \frac{d_p g_{\rm YM}^2 N}{U^{7-p}} \qquad d_p = 2^{7-2p}\pi^{\frac{9-3p}{2}}\Gamma(\frac{7-p}{2}) \tag{8}$$

with $U = r/\alpha'$, r being the radial variable for the transverse directions $x_{p+1}, ..., x_9$. We will consider the field theory limit

$$U = \text{fixed} \qquad g_{\rm YM}^2 = (2\pi)^{p-2} g_s \alpha'^{(p-3)/2} = \text{fixed} \qquad \alpha' \to 0 \tag{9}$$

The parameter U is interpreted, in the gauge theory side, as the energy scale which we want to keep finite.

(a) What does the limit (9) imply on g_s when $p > 3$? What is special about the case $p = 3$?

(b) In the limit (9), show that the metric and the dilaton ϕ reduce to

$$ds^2 = \alpha' \left(\frac{U^{(7-p)/2}}{g_{\rm YM}\sqrt{d_p N}} dx_{||}^2 + \frac{g_{\rm YM}\sqrt{d_p N}}{U^{(7-p)/2}} dU^2 + g_{\rm YM}\sqrt{d_p N} U^{(p-3)/2} d\Omega_{8-p}^2 \right) \tag{10}$$

$$e^{\phi} = (2\pi)^{(2-p)} g_{\rm YM}^2 \left(\frac{g_{\rm YM}^2 d_p N}{U^{7-p}} \right)^{\frac{3-p}{4}} \tag{11}$$

(c) Show that the curvature (Ricci scalar) associated to this metric is given by

$$\alpha' R \sim \frac{1}{g_{\rm eff}} \tag{12}$$

where we defined the effective coupling in the field theory side

$$g_{\rm eff}^2 = \frac{g_{\rm YM}^2 N}{U^{3-p}} \tag{13}$$

(d) The supergravity regime is trustworthy whenever the coupling is small $g_s \sim e^{\phi} \ll 1$ and the curvature is small compared to the string scale $\alpha' R \ll 1$. Show that satisfying both conditions implies

$$1 \ll g_{\rm eff}^2 \ll N^{\frac{4}{7-p}} \tag{14}$$

(e) Rewrite the above condition in terms of U, and distinguish physically each region. In particular, what happens to the dilaton at large radius and at small radius for different values of p ?